# The Polar Equation of the Conic Sections

*Assume that there is an F point and an L line in the plane which does not match the F point. Prove that the geometric location of the points of the P in the plane, for which the quotient of the d(P,F) and d(P,L) distances is equal to an e positive constant, determines a conic section. Prove that the F point is going to be one of the focus points of the conic section received*

The names of these given values are the following:

- F point is the focus point,
- L is the lead line and the directrix and
- the e is the eccentricity of the conic section.

The advantage of this geometric construction against the other description of the conic sections is that it is universal for all the types of the conic sections.

But why are these shapes called conic sections? Because they are created from the cone by splitting it up. In other words, the conic section is the intersection line of the cone and a plane.

Naturally our first step is going to be to illustrate the mathematical concepts of the task. We can use the tubeplot procedure to draw the cone. The tubeplot draws a tube around a 3D [x(t),y(t),z(t)] curve as a symmetry axis and the radius of this tube can change along the curve. The tubeplot can be found in the plots package. Its syntax is:

$$tubeplot([x(t), y(t), z(t), t = a..b, radius = r(t)], opci\acute{o}k),$$

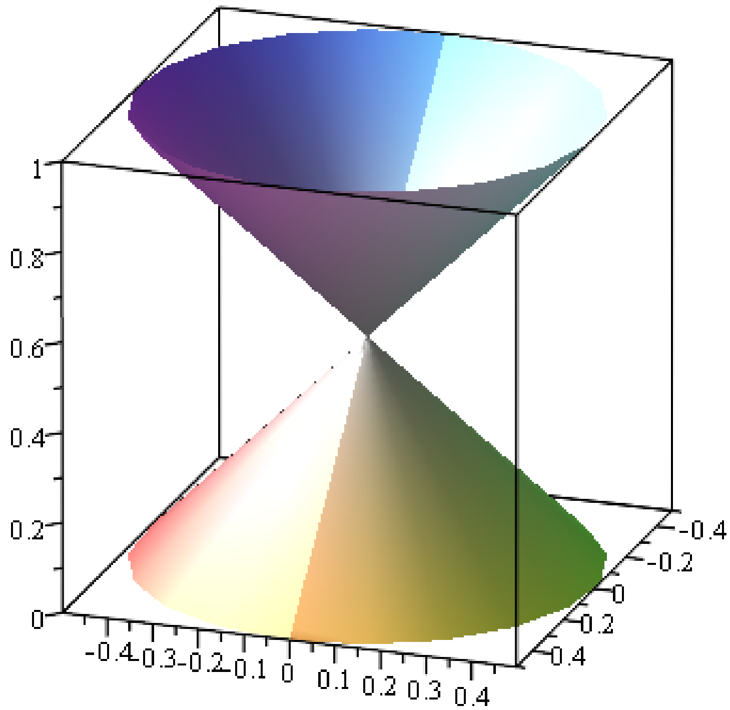
The [x(t),y(t),z(t)] describes the spatial curve while the t parameter changes in the a..b interval. The change of the radius is given by the r(t) expression.

The following command creates a plot object representing such a cone the symmetry axis of which is the z axis and the radius of the tube changes linear with the changing of the z.

```
> with(plots)
Warning, the name changecoords has been redefined
```

```
> kúp := tubeplot([0, 0, t, t = 0..1, radius = 0.5 - t], tubepoints = 20)
```

```
> kúp
```




The general equation of the plane in the spatial [x,y,z] right angle coordinate system is the Ax+By+Cz=D linear equation in which case the [A,B,C] are the normal vector of the plane and the D is a real number. The plane can be drawn by the implicitplot3d procedure located in the plots package. This procedure is the 3D counterpart of the implicitplot procedure. The

$$implicitplot3d(F(x, y, z) = 0, x = a..b, y = c..d, z = e..f, \textit{további opciók});$$

command displays all those [x,y,z] spatial points which fulfil the F(x,y,z)=0 equation between the limits given for the x,y,z coordinates and the a..b,c..d and e..f.

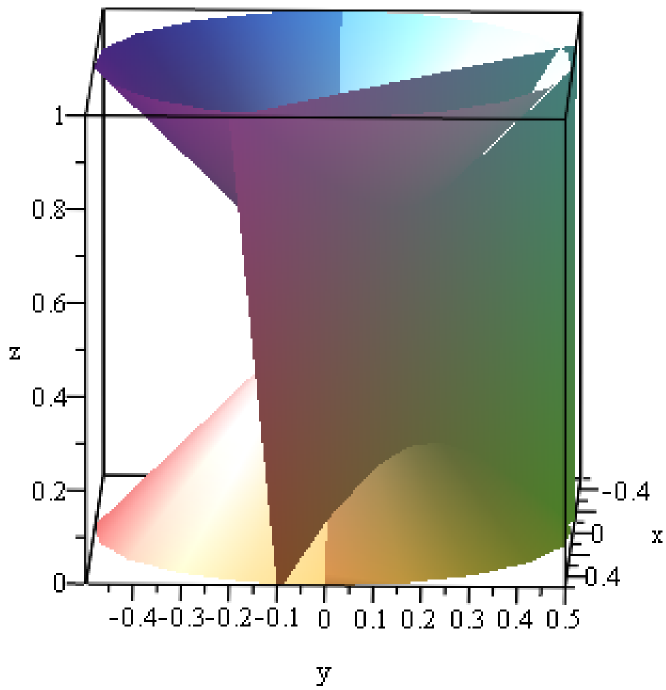
The plane can be situated in three different ways compared to the cone.

If the plane crosses both superficies of the double circular cones then the intersection is a hyperbola.

> $sík_1 := implicitplot3d(x + y + 0.1\,z = 0.4, x = -.5..0.5, y = -.5..0.5, z = 0..1, grid = [8, 8, 16]);$

$$-1$$

> $display3d([kúp, sík_1], style = patch, orientation = [125, 75], scaling = constrained)$

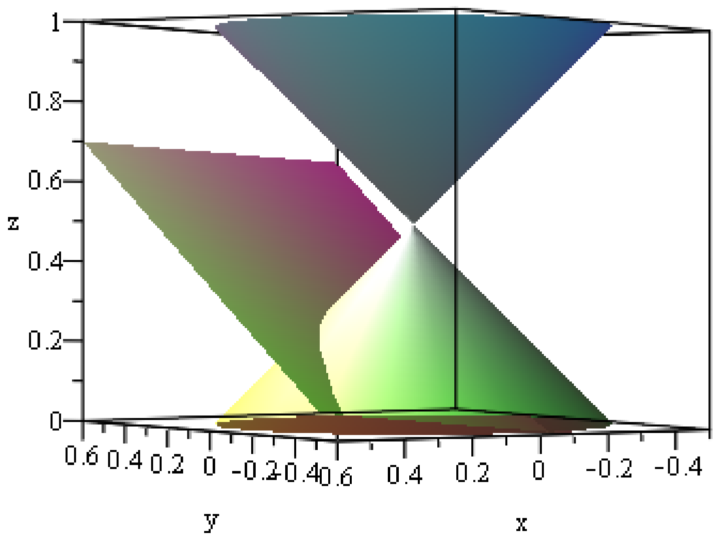



>

If the plane crosses only one superficies of the circular cone but the intersection is not a closed curve, that is, the plane is parallel with one component of the cone then the intersection is a parabola.

> $s\acute{i}k_2 := implicitplot3d(z - x = 0.1, x = -.6..0.6, y = -.6..0.6, z = 0..1, grid = [8, 8, 16]);\ -1$

> $display3d([k\acute{u}p, s\acute{i}k_2], style = patch, orientation = [90, 75], scaling = constrained)$

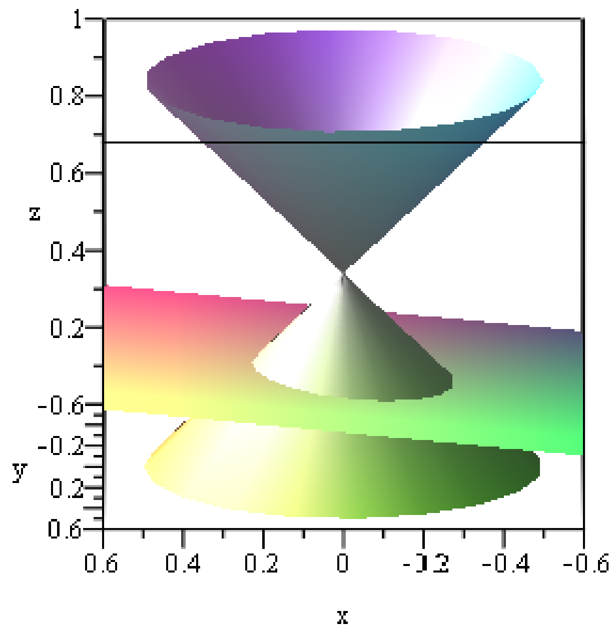



>

Finally, if the plane crosses one superficies of the circular cone, but the intersection is a closed curve, we get an ellipse.

> $sík_3 := implicitplot3d(z - 0.1\,x = 0.25, x = -.6\,..0.6, y = -.6\,..0.6, z = 0\,..1, grid = [8, 8, 16]);\ -1$

> $display3d([kúp, sík_3], style = patch, orientation = [90, 75], scaling = constrained)$




If we enter an appropriate right angle coordinate system in the plane that crosses the cone, then the following equations describe the conic sections in this coordinate system:

$$parabola := x^2 = 4 \cdot p \cdot y$$

$$hiperbola := \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$ellipszis := \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The p, a and b are real parameters (see their meaning in the tasks). These equations are called the centre equation of the appropriate conic section.

After such a long preparation let's get down to the task. Rotate the plane determined by the F and L mentioned in the task in a way that the L line should be the vertical position and the F point should be in the left half plane of the L line.

Introduce such a polar coordinate system to the planes of the F and P the pole of which is the given F point and its polar axis is perpendicular to the L line (see the graph). The intersection of the L line and the polar axis should be the D(d,0) polar coordinated point in which case the $0<d$ is the distance between the F point and the L line.

Assume that the polar coordinate of a P point [r,theta] in the plane is such for which the

$$\frac{d(P,F)}{d(P,L)} = e$$

equation is fulfilled. The Q point should be that point of the L line for which [képlet]. The following relations can be determined for the distance by using the newly introduced notations

> $with(student)$

> $d(P,F) := r$

$$d(P,F) := r \quad (1)$$

> $d(P,Q) := d - d(P,F)\cos(\theta)$

$$d(P,Q) := d - r\cos(\theta) \quad (2)$$

The P point fulfils the assumptions above only if it fulfils the following equation from which we can get the general polar equation of the conic sections by solving the equation for the r.

> $kúp := \frac{d(P,F)}{d(P,Q)} = e$

$$kúp := \frac{r}{d - r\cos(\theta)} = e \quad (3)$$

>

> $kúpszelet := 'r' = normal(solve(kúp, r))$

$$kúpszelet := r = \frac{e\,d}{1 + e\cos(\theta)} \quad (4)$$

We have received the polar equation of the conic section. Admit that it is questionable how an ellipse, parabola or hyperbola can be created from this. But since we use a computer-algebra system, we can easily come over this problem.

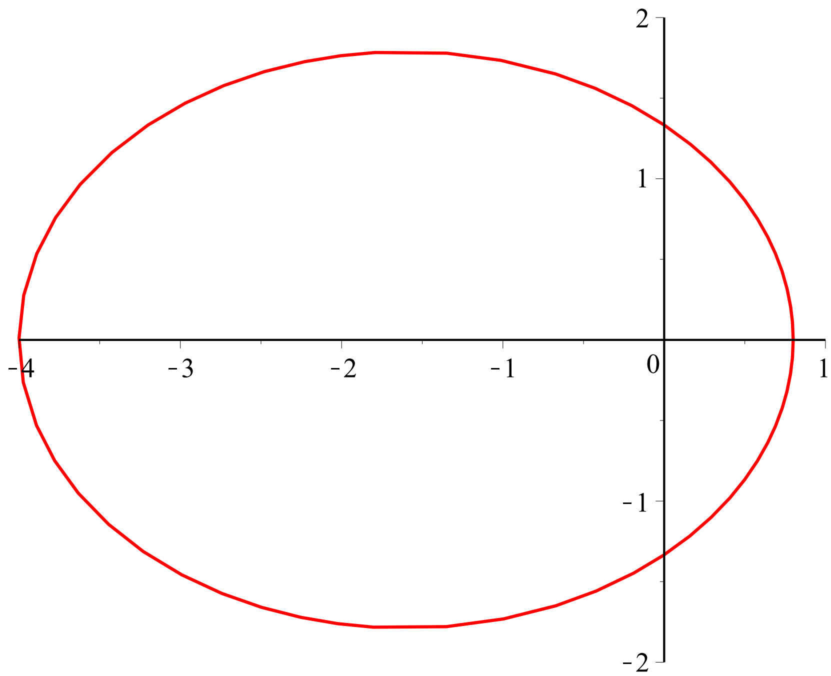
At first, our aim is to gain some experience. We give different values for the e and d parameters in the equation of the conic section and we draw consequences from the graphs created. Let's start with the

$e = \frac{2}{3}$ és $d = 2$ values!

> $r_1 := subs\left(\left\{e = \frac{2}{3}, d = 2\right\}, rhs(kúpszelet)\right)$

$$r_1 := \frac{4}{3}\,\frac{1}{1 + \frac{2}{3}\cos(\theta)} \quad (5)$$

> $plots_{polarplot}([r_1, \theta, \theta = 0..2\,\mathrm{Pi}], -4..1, -2..2, scaling = constrained)$

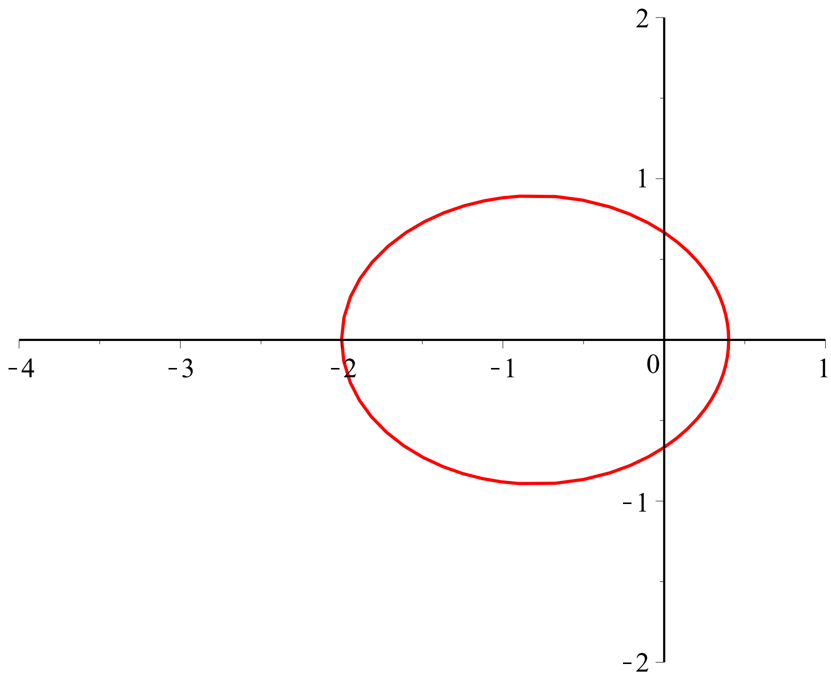



The shape received is an ellipse. First, let’s change the value of the d. Assume thatt $e = \frac{2}{3}$ and $d = 1$.

> $r_1 := subs\left(\left\{d = 1, e = \frac{2}{3}\right\}, rhs(kúpszelet)\right)$

$$r_1 := \frac{2}{3} \frac{1}{1 + \frac{2}{3}\cos(\theta)} \tag{6}$$

> $plots_{polarplot}\left(\left[r_1, \text{theta}, \text{theta} = 0 ..2\, \text{Pi}\right], -4 ..1, -2 ..2, scaling = constrained\right)$



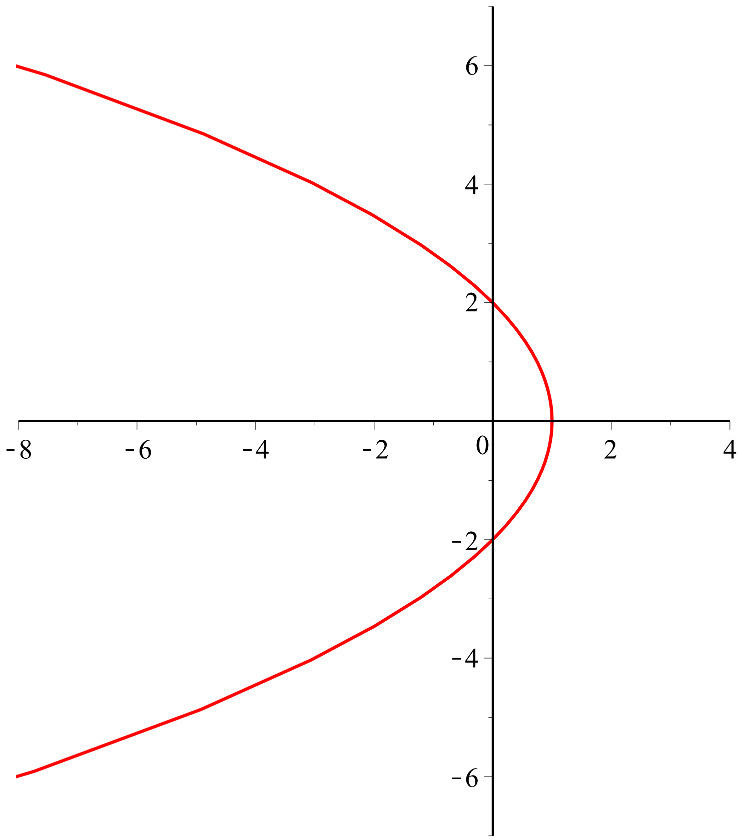

Although the type of the conic section had not changed we got a smaller ellipse. We assume that by increasing the d we are going to get a bigger ellipse. Feel free to try it.

Let's increase the eccentricity of the e. What is going to happen if we choose e=1 and d=2?

> $r_2 := subs(\{e=1, d=2\}, rhs(kúpszelet))$

$$r_2 := \frac{2}{1+\cos(\theta)} \tag{7}$$

> $plots_{polarplot}([r_2, \text{theta}, \text{theta}=0..2\,\text{Pi}], -8..4, -7..7, scaling=constrained)$

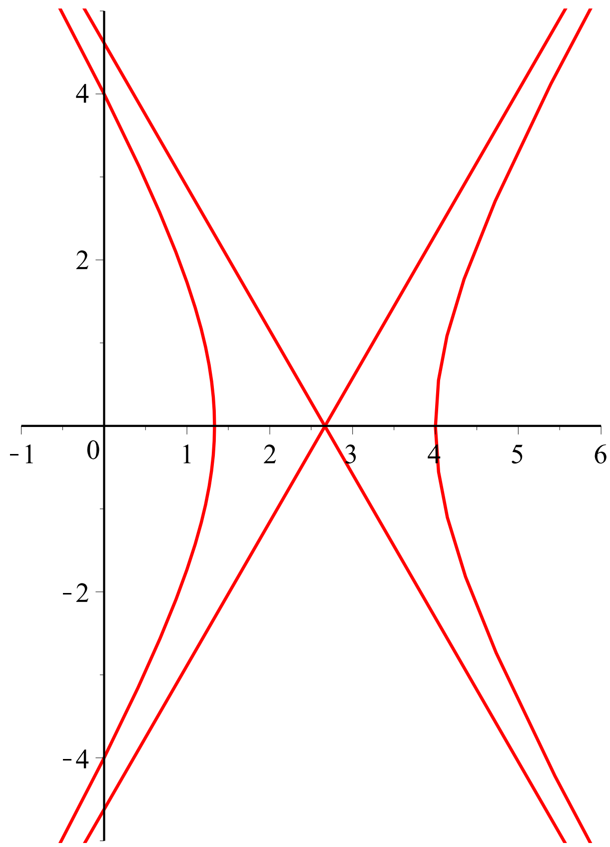



We have received a parabola thus the changing of the eccentricity resulted in another type of conic section. Let's continue with increasing the value of the e to 2

> $r_3 := subs(\{e=2, d=2\}, rhs(kúpszelet))$

$$r_3 := \frac{4}{1+2\cos(\theta)} \tag{8}$$

> $plots_{polarplot}([r_3, \text{theta}, \text{theta}=0..2\,\text{Pi}], -1..6, -5..5, scaling=constrained)$




Great! We have got the third type of conic section, namely the hyperbola. We can draw the first conclusion that the type of the conic section depends on the eccentricity of the e.

Instead of the task in question we are going to show only that in the case of 0<e<1, the (4) polar equation determines an ellipse. The reader should solve the e=1 and the e>1 cases.

First, let's multiply the general polar equation of the conic section with its denominator. This can be easily executed since the

$1 + e\cos(\text{theta})$ expression cannot be 0 because due to the it is

$0 < e < 1$ és $-1 < cos(\theta) < 1$ miatt $e \cdot cos(\theta) > -\mathrm{e} > -1$!

> *ellipszis* := *expand*(*denom*(*rhs*(*kúpszelet*)) *kúpszelet*)

$$ellipszis := r + r\, e \cos(\theta) = e\, d \qquad \textbf{(9)}$$

We want to get the following equation of the ellipse in a right angle coordinate system

$$\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} = 1$$

Because of this, let's convert the polar equation above into an equation given in right angle coordinates using the transitional formulas studied in chapter 4.2. Execute the two substitutions with one powsubs procedure

> $powsubs\left(r\cos(\theta)=x,\ r=\sqrt{x^2+y^2},\ ellipszis\right)$

$$\sqrt{x^2+y^2}+e\,x=e\,d \quad \textbf{(10)}$$

Let's express the square root of the x2+y2 with the isolate procedure from the equation received then square both sides of the equation. To do this, use the map procedure by giving the t->t2 function as a first parameter. Then expand the expression created with the expand procedure on the right side of the equation.

> $isolate\left(\%,\sqrt{x^2+y^2}\right)$

$$\sqrt{x^2+y^2}=e\,d-e\,x \quad \textbf{(11)}$$

> $map(t\rightarrow t^2,\%)$

$$x^2+y^2=(e\,d-e\,x)^2 \quad \textbf{(12)}$$

> $expand(\%)$

$$x^2+y^2=e^2\,d^2-2\,e^2\,d\,x+e^2\,x^2 \quad \textbf{(13)}$$

Figyeljük meg, hogy az **isolate**, a **map** és az **expand** eljárások milyen szépen egymásba illeszthetk!

Az $x$ változó a kapott egyenlet jobb oldalán szerepl második és harmadik tagban is elfordul. Rendezzük át az egyenletet úgy, hogy ezek kerüljenek át az egyenlet bal oldalára! Ahhoz, hogy ne kelljen a megfelel tagokat még egyszer begépelni, alkalmazzuk az egyenlet jobb oldalára az **op** eljárást. Mivel az egyenlet jobb oldala összeg típusú, ezért az operanduszok a megfelel tagok lesznek. Az els operandusz az els tag, a második operandusz a második tag, stb.

> $tagok := op(2, rhs(\textbf{(13)}))+op(3, rhs(\textbf{(13)}))$

$$tagok := -2\,e^2\,d\,x+e^2\,x^2 \quad \textbf{(14)}$$

> $\textbf{(13)} - tagok$

$$x^2+y^2+2\,e^2\,d\,x-e^2\,x^2=e^2\,d^2 \quad \textbf{(15)}$$

Divide the equation received by $1-e^2$-tel, then convert it to a complete square in the x.

> $expand\left(\frac{\%}{1-e^2}\right)$

$$\frac{x^2}{1-e^2}+\frac{y^2}{1-e^2}+\frac{2\,e^2\,d\,x}{1-e^2}-\frac{e^2\,x^2}{1-e^2}=\frac{e^2\,d^2}{1-e^2} \quad \textbf{(16)}$$

> $completesquare(\%, x)$

$$\left(x-\frac{e^2\,d}{-1+e^2}\right)^2-\frac{e^4\,d^2}{(-1+e^2)^2}+\frac{y^2}{1-e^2}=\frac{e^2\,d^2}{1-e^2} \quad \textbf{(17)}$$

The constant term appearing on the left side of the equation, which contains neither an x nor a y, has to be put to the right side of the equation. Finally, we should find the common denominator of the two expressions appearing on the right side of the equation with the normal procedure. Notice that the normal procedure could not have been applied for both sides of the equation because the result of the complete square would have been wrong.

> $konstans := op(2, lhs(\textbf{(17)}))$

$$konstans := -\frac{e^4 d^2}{(-1+e^2)^2} \quad \textbf{(18)}$$

> **(17)** $- konstans$

$$\left(x - \frac{e^2 d}{-1+e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2 d^2}{1-e^2} + \frac{e^4 d^2}{(-1+e^2)^2} \quad \textbf{(19)}$$

> $lhs(\%) = normal(rhs(\%))$

$$\left(x - \frac{e^2 d}{-1+e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2 d^2}{(-1+e^2)^2} \quad \textbf{(20)}$$

We can see on this equation that it describes an ellipse because the coefficients of the complete squares are positive. The $1-e^2$

expression located in the denominator is positive because of the 0<e<1. But we are not satisfied with this because we want to get the centre equation of the ellipse.

Let's denote the amount extracted from the x within the complete square with the u and substitute it into the equation with the help of the subs procedure

> $ellipszis := subs\left(\frac{e^2 d}{-1+e^2} = u, \textbf{(20)}\right)$

$$ellipszis := (x-u)^2 + \frac{y^2}{1-e^2} = \frac{e^2 d^2}{(-1+e^2)^2} \quad \textbf{(21)}$$

We want the right side of the equation be normed to 1. Let's freeze the x-u sub expression so that the expand procedure to be used would not spoil the $(x-u)^2$ expression. At the same time, the notation p is introduced instead of the $1-e^2$ value. After the freezing, feel free to do the division and the simplifications needed without spoiling the complete squares.

> $fagyaszt := x - u = z$

$$fagyaszt := x - u = z \quad \textbf{(22)}$$

> $powsubs(1 - e^2 = p, fagyaszt, ellipszis)$

$$z^2 + \frac{y^2}{p} = \frac{e^2 d^2}{p^2} \quad \textbf{(23)}$$

> $expand\left(\frac{\% p^2}{e^2 d^2}\right)$

$$\frac{p^2 z^2}{e^2 d^2} + \frac{p y^2}{e^2 d^2} = 1 \quad \textbf{(24)}$$

Finally, let's defrost it. In order to get the usual equation of the ellipse, the following substitutions have to be executed.

> $helyettesítés := \frac{p^2}{e^2 d^2} = \frac{1}{a^2}, \frac{p}{e^2 d^2} = \frac{1}{b^2}$

**(25)**

$$helyettesítés := \frac{p^2}{e^2 d^2} = \frac{1}{a^2}, \frac{p}{e^2 d^2} = \frac{1}{b^2} \quad \mathbf{(25)}$$

> $felenged := z = x - u$

$$felenged := z = x - u \quad \mathbf{(26)}$$

> $powsubs\left(helyettesítés, felenged, \frac{p^2 z^2}{e^2 d^2} + \frac{p y^2}{e^2 d^2} = 1\right)$

$$\frac{(-x + u)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \mathbf{(27)}$$

Summarising what has been said, we have got the equation of an ellipse for which the length of the half axes is

> $a := \frac{e\, d}{1 - e^2}; b := \frac{e\, d}{\sqrt{1 - e^2}}$

$$a := \frac{e\, d}{1 - e^2}$$
$$b := \frac{e\, d}{\sqrt{1 - e^2}} \quad \mathbf{(28)}$$

and the coordinates of the C symmetry centre of the ellipse are [u,0] in which case

> $u := \frac{d\, e^2}{e^2 - 1}$

$$u := \frac{d\, e^2}{-1 + e^2} \quad \mathbf{(29)}$$

We also have to prove that the F point is one of the focus points of the ellipse. Denote the distance between the symmetry centre and the focus point of the ellipse with the usual c. It is known that $c^2 + b^2 = a^2$. Let's calculate the value of the c.

> $c := simplify\left(\sqrt{a^2 - b^2}, assume = positive\right)$

$$c := \frac{e^2 d}{|-1 + e^2|} \quad \mathbf{(30)}$$

We were able to do the simplification partly because the system does not know the sign of the $-1 + e^2$ expression. How could Maple know that $0 < e < 1$ if we do not inform it about this? To do this, use the assume and additionally procedures.

> $assume(0 < e); 1; additionally(e < 1); 1$

> $about(e); 1$

```
Originally e, renamed e~:
  is assumed to be: RealRange(Open(0),Open(1))
```

> $simplify(c); 1$

$$-\frac{e\sim^2 d}{-1+e\sim^2} \quad (31)$$

What has happened? First, we assumed that 0<e<1 with the help of the assume procedure then we gave the e<1 as another piece of additional information by using the additionally procedure. Then we asked with the help of the about procedure what Maple knew about the variable e. It responded that it had renamed the variable e to e~ a and it assumed that the e was in the open (0,1) interval.

Let's continue the calculations. If we compare the formulas received for the u and c we can see that c=-h. This equality proves that the F point is one of the focus points of the ellipse because the right angle coordinates of the C symmetry centre of the ellipse are [-u,0]. However, the d(F,C)=c distance is equal to the (-u) so the F point is at a focus length from the centre C. Finally, the c=-h equality can be easily understood with the help of Maple.

> $c+u=simplify(c+u);$

$$-\frac{e\sim^2 d}{-1+e\sim^2}+u=\frac{-e\sim^2 d-u+u\,e\sim^2}{-1+e\sim^2} \quad (32)$$

>

We have solved the task.

# What Have You Learnt About Maple?

• The tubeplot procedure of the plots package draws a tube around a $[x(t), y(t), z(t)]$ plane curve given parametrically. The radius of the tube changes by the r=r(t) function. The t parameter draws the tube from the value of the a to the b. Its syntax is:

$$tubeplot([x(t), y(t), z(t), t=a..b, radius=r(t)], tubepoints=n).$$

The system compiles the cross section of the tube from the regular n angle.

• The implicitplot3d procedure of the plots package draws the [x,y,z] spatial points in 3-D fulfilling the $F(x, y, z)=0$ equation. Its syntax is:

$$implicitplot3d(F(x, y, z)=0, x=a..b, y=c..d, z=e..f, grid=[n, m, k], orientation=[\mathrm{theta}, \mathrm{phi}]),$$

It uses the digits given in the grid option to scale the x,y,z axes and the $[a, b]$, $[c, d]$ and $[e, f]$ intervals. The theta and phi angles give the direction of the projection and the viewpoint. Theta gives it in the angles by the x axis and phi gives it in the angles by the z axis.

• The assume and assume (x, property) instructions are used to make assumptions for the variables for Maple. The assumption is usually given in the form of an equality or equation.

These are most common cases for the properties of the x Maple object:

ointeger

oreal

ocomplex

ocontinuous

The instruction makes Maple rename the variables of the task by indicating a so-called tilde ~ sign subsequent to the name. This means that we previously made an assumption on the variable. The unassign(x) procedure frees these boundaries concerning the variable x.

• With the help of the about(x) instructions the constraints and the assumptions made on the variable x can be listed.

• More than one assumption for a variable can be determined by the additionally instruction. While every assumption given in every newer assume instruction overwrites the previously given assumptions provided that they refer to the same variable, we can call arbitrary numbers of assumptions from the additionally procedure. The assumptions are concatenated to the assumptions previously specified.

• The is(assumption) or isgiven(x,property) check if the given assumption comes true and if the x object has its property in question

# Exercises

1. Prove that in the case of e=1 we get a parabola.

2. Prove that in the case of e>1 we get a hyperbola.

3. Can we get a circle with the help of the procedure given in the task? (In the case of a circle, the length of the half axes are equal, that is, a=b=R.)

4. Prove that if the F point lies on the right side half plane determined by the L line then we get a conic section the polar equation of which is: [képlet]

5. Prove that if the L line is horizontal and we enter the polar axis parallel with the L then we get a conic section the polar equation of which is

$$r = \frac{d\,e}{1 + e\,sin(\theta)} \text{ or } r = \frac{d\,e}{1 - e\,sin(\theta)}$$

depending on in which half plane the F point lies.

6. Draw the appropriate curves based on the polar equations of the given conic sections.

| | | |
|---|---|---|
| $r = \frac{6}{3 + \sin(\text{theta})}$ | $r = \frac{4}{\cos(\text{theta}) - 2}$ | $r = \frac{15}{4 - 4\cos(\text{theta})}$ |
| $r = \frac{10}{3\sin(\text{theta}) + 2}$ | $r = \frac{3}{2 - 2\sin(\text{theta})}$ | $r = \frac{3}{2 + 2\cos(\text{theta})}$ |

7. Assume that the ellipse is the geometric location of P points in the plane. The sum of the distances of these P points measured from two given points is equal to a given distance, that is, for which

$d(P, F1) + d(P, F2) = 2\,a$

in which case the F1 and F2 are the two points and 0<a. Prove with the help of Maple that in an appropriate right angle coordinate system the

$\frac{x-h}{a^2} + \frac{y-k}{b^2} = 1$ equation fulfils the [x,y] points of the ellipse in which case $b^2 = a^2 - c^2$ and $c = \frac{d(F1, F2)}{2}$.

8. Assume that the hyperbola is the geometric location of P points in the plane. The difference of the distances of these P points measured from two given points is equal to a given distance, that is, for which

$$d(P, F1) - d(P, F2) = 2\,a$$

in which case the F1 and F2 are the two points and 0<a. Prove with the help of Maple that in an appropriate right angle coordinate system the

$\frac{x-h}{a^2} + \frac{y-k}{b^2} = 1$ equation fulfils the [x,y] points of the hyperbola in which case $b^2 + a^2 = c^2$ és $c = \frac{d(F1, F2)}{2}$

9. Prove that the equation of the parabola in a right angle coordinate system is

$x^2 = 4\,p\,y$ in which case 2p=d(F,L).